Introductory Mathematics
Illustrative Problems and Solutions
(Part 3 of 3)

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16. Arithmetic and geometric sequences

Arithmetic and geometric sequences – example 1

Question

Let n be the number of terms in a sequence of consecutive even numbers starting at 14 whose sum is 558. Then $n =?$

The first n even number starting from 14 are:

$$14,\ 16,\ 18,\ \ldots,\ 14+2(n-1)$$

This is an arithmetic sequence, where the first term $a = 14$, with common difference $d = 2$, and the number of terms is n. The sum of those n numbers is given by

$$S = \frac{n(2a+(n-1)d)}{2} = \frac{n(28+2(n-1))}{2} = n(n+13) = 558$$

We have $n = 18$.

Arithmetic and geometric sequences – example 2

Question

Let n be the number of terms in a sequence of consecutive odd numbers starting from 15, whose sum is 680. What is the largest number in the sequence?

The first n odd numbers starting from 15 are:

$$15,\ 17,\ 19,\ \ldots,\ 15+2(n-1)$$

This is an arithmetic sequence with first term $a = 15$, common difference $d = 2$, and n terms. The sum is:

$$S = \frac{n(2a+(n-1)d)}{2} = \frac{n(30+2(n-1))}{2} = n(n+14) = 680$$

We get $n = 20$, so the largest term is:

$$15+2(n-1) = 15+38 = 53$$

Arithmetic and geometric sequences – example 3

Question

Insert numbers between 4 and 37 so that together they form an arithmetic sequence with total sum 246. What is the common difference of this sequence?

Let the arithmetic sequence start at 4, end at 37, and have n terms. The sum of the sequence is given by

$$S = \frac{n(4+37)}{2} = 246 \implies n = 12$$

Hence the common difference is:

$$d = \frac{37-4}{11} = \frac{33}{11} = 3.$$

Arithmetic and geometric sequences – example 4

Question

The sum of the first three terms of a geometric sequence is -12. The difference between the fourth and the first term is 36. What is the sum of the second and fourth term?

Let the first term be a, and the common ratio be r. Then

$$a + ar + ar^2 = -12 \implies a(1 + r + r^2) = -12 \tag{1}$$

$$ar^3 - a = 36 \implies a(r^3 - 1) = 36 \tag{2}$$

Divide (2) by (1):

$$\frac{r^3 - 1}{1 + r + r^2} = -3 \implies \frac{(r-1)(r^2 + r + 1)}{1 + r + r^2} = -3 \implies r = -2$$

Substitute back into (1) gives $a = -4$.

$$ar + ar^3 = 8 + 32 = 40$$

Arithmetic and geometric sequences – example 5

Question

If we add the same number to each of the numbers 2, 7, and 17, we get the first three terms of a geometric sequence. What is the value of the sixth term of this geometric sequence?

Let the number added to each be x. Then the sequence becomes:

$$2 + x, \; 7 + x, \; 17 + x$$

These must form a geometric sequence, so:

$$(7 + x)^2 = (2 + x)(17 + x) \implies x = 3$$

Now the geometric sequence is:

$$5, \; 10, \; 20, \; \cdots \implies \text{common ratio } r = 2$$

Sixth term:

$$ar^5 = 5 \times 2^5 = 5 \times 32 = 160$$

Arithmetic and geometric sequences – example 6

Question

If we add the same number to each of the numbers 7, 15, and 27, we get the first three terms of a geometric sequence. What is the value of the fifth term of this geometric sequence?

Let the number added be x. Then the sequence becomes:

$$7 + x,\ 15 + x,\ 27 + x$$

These must form a geometric sequence:

$$(15 + x)^2 = (7 + x)(27 + x) \implies x = 9$$

The sequence is:

$$16,\ 24,\ 36,\ \ldots \quad \implies \text{common ratio } r = \frac{24}{16} = 1.5$$

Fifth term:

$$ar^4 = 16 \times (1.5)^4 = 81$$

Arithmetic and geometric sequences – example 7

Question

If we add the same number to each of the values 2, 18, and 42, we get the first three terms of a geometric sequence. What are the first 6 terms of this geometric sequence?

Let the number added be x. Then the sequence becomes:

$$2 + x,\ 18 + x,\ 42 + x$$

Since it's geometric:

$$(18 + x)^2 = (2 + x)(42 + x) \implies x = 30$$

The common ratio is

$$r = \frac{48}{32} = \frac{3}{2}$$

The first 6 terms are

$$32,\ 48,\ 72,\ 108,\ 162,\ 243$$

Arithmetic and geometric sequences – example 8

Question

If we add the same number to each of the values 3, 13, and 33, we get the first three terms of a geometric sequence. What are the first 6 terms of this geometric sequence?

Let the number added be x. Then the sequence becomes:

$$3 + x,\ 13 + x,\ 33 + x$$

Since it's geometric:

$$(13 + x)^2 = (3 + x)(33 + x) \implies x = 7$$

The common ratio is:

$$r = \frac{20}{10} = 2$$

The first 6 terms are:

$$10,\ 20,\ 40,\ 80,\ 160,\ 320$$

Arithmetic and geometric sequences – example 9

Question

The sum of the first three terms of a geometric sequence is -21. The difference between the fourth and first term is -21. What are the first four terms of the sequence?

Let the first term be a, and the ratio r. Then

$$a + ar + ar^2 = -21 \Longrightarrow a(1 + r + r^2) = -21 \quad (1)$$

$$ar^3 - a = -21 \Longrightarrow a(r^3 - 1) = -21 \quad (2)$$

Divide (2) by (1):

$$\frac{(r-1)(1+r+r^2)}{1+r+r^2} = 1 \Longrightarrow r = 2$$

Substitute to (1) gives $a = -3$. The first four terms of the sequence are

$$-3,\ -6,\ -12,\ -24$$

Arithmetic and geometric sequences – example 10

Question

The sum of the first three terms of a geometric sequence is -14. The difference between the fourth and first term is also -14. What are the first four terms of the sequence?

Let the first term be a and common ratio be r. Then

$$a + ar + ar^2 = -14 \implies a(1 + r + r^2) = -14 \tag{1}$$

$$ar^3 - a = -14 \implies a(r^3 - 1) = -14 \tag{2}$$

Divide (2) by (1):

$$\frac{(r-1)(1+r+r^2)}{1+r+r^2} = 1 \implies r = 2$$

Substitute to (1) gives $a = -2$. The first four terms of the sequence are

$$-2,\ -4,\ -8,\ -16$$

Arithmetic and geometric sequences – example 11

Question

The sum of the first three terms of a geometric sequence is equal to -28. The difference between the fourth and the first term of the sequence is also -28. What are the first four terms of the sequence?

Same as before, we get $r = 2$ and

$$7a = -28 \Longrightarrow a = -4$$

The first four terms are

$$-4, \ -8, \ -16, \ -32$$

Arithmetic and geometric sequences – example 12

Question

The first four terms of an arithmetic sequence are denoted by a_1, a_2, a_3, a_4. It is given that:

$$a_1 + a_3 = 24, \quad a_2 - a_4 = -6$$

What are $a_1,\ a_2,\ a_3$, and a_4?

Let the first term be a and common difference d. Then:

$$a_1 = a, \quad a_2 = a + d, \quad a_3 = a + 2d, \quad a_4 = a + 3d$$

And

$$a_1 + a_3 = a + (a + 2d) = 2a + 2d = 24 \Longrightarrow a + d = 12 \quad (1)$$

$$a_2 - a_4 = (a + d) - (a + 3d) = -2d = -6 \Longrightarrow d = 3 \quad (2)$$

Substitute $d = 3$ into (1) gives $a = 9$. The four values are

$$9,\ 12,\ 15,\ 18$$

Arithmetic and geometric sequences – example 13

Question

The first four terms of an arithmetic sequence are denoted by a_1, a_2, a_3, a_4. It is given that:

$$a_1 + a_4 = 25, \quad a_1 + a_3 = 22$$

What are $a_1,\ a_2,\ a_3$, and a_4?

Let the first term be a and common difference d. Then

$$a_1 = a, \quad a_2 = a + d, \quad a_3 = a + 2d, \quad a_4 = a + 3d$$

$$a + (a + 3d) = 2a + 3d = 25$$
$$a + (a + 2d) = 2a + 2d = 22$$

We have $d = 3, a = 8$ and the four values are

$$8,\ 11,\ 14,\ 17$$

Arithmetic and geometric sequences – example 14

Question

The first four terms of an arithmetic sequence are denoted by a_1, a_2, a_3, a_4. It is given that:

$$a_1 + a_2 + a_3 = 33, \quad a_4 - a_2 = 6$$

What are $a_1,\ a_2,\ a_3$, and a_4?

Let the first term be a and common difference d. Then

$$a_1 = a, \quad a_2 = a + d, \quad a_3 = a + 2d, \quad a_4 = a + 3d$$

$$\begin{aligned} a_1 + a_2 + a_3 &= a + (a + d) + (a + 2d) = 3a + 3d = 33 \Longrightarrow a + d = 11 \\ a_4 - a_2 &= (a + 3d) - (a + d) = 2d = 6 \Longrightarrow d = 3 \end{aligned}$$

The four values are

$$8,\ 11,\ 14,\ 17$$

Arithmetic and geometric sequences – example 15

Question

The first four terms of an arithmetic sequence are denoted by a_1, a_2, a_3, a_4. It is given that:

$$a_1 + a_2 + a_3 = 33, \quad a_4 - a_2 = 4$$

What are $a_1,\ a_2,\ a_3$, and a_4?

Let the first term be a and common difference d. Then

$$a_1 = a, \quad a_2 = a + d, \quad a_3 = a + 2d, \quad a_4 = a + 3d$$

$$a_1 + a_2 + a_3 = a + (a + d) + (a + 2d) = 3a + 3d = 33 \Longrightarrow a + d = 11$$
$$a_4 - a_2 = (a + 3d) - (a + d) = 2d = 4 \Longrightarrow d = 2$$

The four values are

$$9,\ 11,\ 13,\ 15$$

17. Analytic geometry – equations of lines and circles

Equations of lines and circles – example 1

Question

What is the general equation of the line that passes through the point $(3,\ -4)$ and is perpendicular to the line $2x - 3y + 6 = 0$?

The slope of the given line is $\frac{2}{3}$. The slope of the perpendicular line is the negative reciprocal $-\frac{3}{2}$. Using the point-slope form:

$$y + 4 = -\frac{3}{2}(x - 3) \Longrightarrow 3x + 2y - 1 = 0$$

Equations of lines and circles – example 2

Question

What is the general equation of the line that passes through the point $(1,\ -3)$ and is perpendicular to the line $4x - 3y + 6 = 0$?

The slope of the given line is $\frac{4}{3}$. The slope of the perpendicular line is the negative reciprocal $-\frac{3}{4}$. Using the point-slope form:

$$y + 3 = -\frac{3}{4}(x - 1) \Longrightarrow 4y + 12 = -3x + 3 \Longrightarrow 3x + 4y + 9 = 0$$

Equations of lines and circles – example 3

Question

What is the general equation of the line that passes through the point $(3,\ -4)$ and is parallel to the line $2x - 3y + 6 = 0$?

The slope of the given line is $\frac{2}{3}$, so the parallel line has the same slope. Using point-slope form:

$$y + 4 = \frac{2}{3}(x - 3) \implies 3y + 12 = 2x - 6 \implies 2x - 3y - 18 = 0$$

Equations of lines and circles – example 4

Question

What is the general equation of the line that passes through the point $(1,\ -3)$ and is parallel to the line $4x - 3y + 6 = 0$?

The given line has slope $\frac{4}{3}$, so the parallel line has the same slope. Using point-slope form:

$$y + 3 = \frac{4}{3}(x - 1) \Longrightarrow 3y + 9 = 4x - 4 \Longrightarrow 4x - 3y - 13 = 0$$

Equations of lines and circles – example 5

Question

Convert the general equation of the circle $x^2 + y^2 - 4x - 12 = 0$ into its standard form. Using the standard form, determine the fundamental characteristics of the circle. Determine the center and radius of the circle. Does the point $(-2,\ 0)$ lie on the circle?

Group terms:

$$(x^2 - 4x) + y^2 = 12$$

Complete the square:

$$(x^2 - 4x + 4) + y^2 = 12 + 4 \Longrightarrow (x - 2)^2 + y^2 = 16$$

So the center is $(2,\ 0)$, and radius is 4.
Since the distance between $(-2,\ 0)$ and $(2,\ 0)$ is 4, the point $(2,\ 0)$ lies on the circle.

Equations of lines and circles – example 6

Question

Convert the general equation of the circle $x^2 + y^2 - 6y - 27 = 0$ into its standard form. Using the standard form, determine the fundamental characteristics of the circle. Determine the center and radius of the circle. Does the point $(0, \ -3)$ lie on the circle?

Group terms:

$$x^2 + (y^2 - 6y) = 27$$

Complete the square:

$$x^2 + \left(y^2 - 6y + 9\right) = 27 + 9 \Longrightarrow x^2 + (y - 3)^2 = 36$$

So the center is $(0, \ 3)$, and radius is 6. Since the distance between $(0, \ -3)$ and $(0, \ 3)$ is 6, the point $(0, \ -3)$ lies on the circle.

Equations of lines and circles – example 7

Question

Convert the general equation of the circle $x^2 + y^2 - 8x - 33 = 0$ into its standard form. Using the standard form, determine the fundamental characteristics of the circle. Determine the center and radius of the circle. Does the point $(11,\ 0)$ lie on the circle?

Group terms:

$$(x^2 - 8x) + y^2 = 33$$

Complete the square:

$$(x^2 - 8x + 16) + y^2 = 33 + 16 \Longrightarrow (x-4)^2 + y^2 = 49$$

So the center is $(4,\ 0)$, and radius is 7. Since the distance between $(4,\ 0)$ and $(11,\ 0)$ is 7, the point $(11,\ 0)$ lies on the circle.

Equations of lines and circles – example 8

Question

Convert the general equation of the circle $x^2 + y^2 - 10y - 11 = 0$ into its standard form. Using the standard form, determine the fundamental characteristics of the circle. Determine the center and radius of the circle. Does the point $(0,\ 11)$ lie on the circle?

Group terms:

$$x^2 + (y^2 - 10y) = 11$$

Complete the square:

$$x^2 + (y^2 - 10y + 25) = 11 + 25 \Longrightarrow x^2 + (y - 5)^2 = 36$$

So the center is $(0,\ 5)$, and radius is 6. Since the distance between $(0,\ 11)$ and $(0,\ 5)$ is 6, the point $(0,\ 11)$ lies on the circle.

Equations of lines and circles – example 9

Question

Convert the general equation of the circle $x^2 + y^2 - 12x - 13 = 0$ into its standard form. Using the standard form, determine the fundamental characteristics of the circle. Determine the center and radius of the circle. Does the point $(0,\ -6)$ lie on the circle?

Group terms:

$$(x^2 - 12x) + y^2 = 13$$

Complete the square:

$$(x^2 - 12x + 36) + y^2 = 13 + 36 \implies (x-6)^2 + y^2 = 49$$

So the center is $(6,\ 0)$, and radius is 7. Since the distance between $(0,\ -6)$ and the center is not 7, the point $(0,\ -6)$ does not lie on the circle.

Equations of lines and circles – example 10

Question

Convert the general equation of the circle $x^2 + y^2 - 4y - 12 = 0$ into its standard form. Using the standard form, determine the fundamental characteristics of the circle. Determine the center and radius of the circle. Does the point $(0, \ -2)$ lie on the circle?

Group terms:

$$x^2 + (y^2 - 4y) = 12$$

Complete the square:

$$x^2 + (y^2 - 4y + 4) = 12 + 4 \Longrightarrow x^2 + (y - 2)^2 = 16$$

So the center is $(0, \ 2)$, and radius is 4. Since the distance between $(0, \ -2)$ and the center is 4, the point $(0, \ -2)$ lies on the circle.

Equations of lines and circles – example 11

Question

What is the general equation of the line that passes through the point $(3,\ 3)$ and is perpendicular to the line $2x - y - 1 = 0$?

The slope of the given line is 2. The slope of the perpendicular line is the negative reciprocal $-\frac{1}{2}$. Using the point-slope form:

$$y - 3 = -\frac{1}{2}(x - 3) \implies x + 2y - 9 = 0$$

Equations of lines and circles – example 12

Question

What is the general equation of the line that passes through the point $(1,\ 1)$ and is perpendicular to the line $x - 3y + 6 = 0$?

The slope of the given line is $\frac{1}{3}$. The slope of the perpendicular line is the negative reciprocal 3. Using the point-slope form:

$$y - 1 = 3(x - 1) \Longrightarrow 3x - y - 4 = 0$$

Equations of lines and circles – example 13

Question

What is the general equation of the line that passes through the point $(3,\ 4)$ and is parallel to the line $2x - 3y - 6 = 0$?

The slope of the given line is $\frac{2}{3}$. The slope of the parallel line is the same. Using the point-slope form:

$$y - 4 = \frac{2}{3}(x - 3) \Longrightarrow 2x - 3y + 6 = 0$$

Equations of lines and circles – example 14

Question

What is the general equation of the line that passes through the point $(1,\ -3)$ and is parallel to the line $4x - 3y + 6 = 0$?

The slope of the given line is $\frac{4}{3}$. The slope of the parallel line is the same. Using the point-slope form:

$$y + 3 = \frac{4}{3}(x - 1) \implies 4x - 3y - 13 = 0$$

Equations of lines and circles – example 15

Question

Convert the general equation of the circle $x^2 + y^2 - 6x + 8y + 16 = 0$ into its standard form. Using the standard form, determine the fundamental characteristics of the circle. Determine the center and radius of the circle. Does the circle intersect with the $x-$axis?

Group terms:

$$x^2 - 6x + y^2 + 8y + 16 = 0$$

Complete the square:

$$(x^2 - 6x + 9) + (y^2 + 8y + 16) = -16 + 9 + 16 \Longrightarrow (x-3)^2 + (y+4)^2 = 9$$

So the center is $(3,\ -4)$, and radius is 3. Set $y = 0$, we get

$$(x-3)^2 = 9 - 16 = -7$$

has no real solution. Thus the circle does not intersect with the $x-$aixs.

Equations of lines and circles – example 16

Question

Convert the general equation of the circle $x^2 + y^2 - 6x - 2y - 26 = 0$ into its standard form. Using the standard form, determine the fundamental characteristics of the circle. Determine the center and radius of the circle. Does the point $(-3,\ 1)$ lie on the circle?

Group terms:

$$(x^2 - 6x) + (y^2 - 2y) = 26$$

Complete the square:

$$(x-3)^2 + (y-1)^2 = 26 + 9 + 1 \implies (x-3)^2 + (y-1)^2 = 36$$

So the center is $(3,\ 1)$, and radius is 6. The distance between the point $(-3,\ 1)$ and the center is equal to the radius. Hence the point $(-3,\ 1)$ lies on the circle.

Equations of lines and circles – example 17

Question

Determine the general equation of the line that passes through the point $(3,\ 3)$ and is perpendicular to the line passing through points $(1,\ 1)$ and $(2,\ 3)$.

Slope of the line through points $(1,\ 1)$ and $(2,\ 3)$ is

$$\frac{3-1}{2-1} = 2$$

Slope of the perpendicular line is $-\frac{1}{2}$. Using point-slope form:

$$y - 3 = -\frac{1}{2}(x-3) \implies x + 2y - 9 = 0$$

Equations of lines and circles – example 18

Question

Determine the general equation of the line that passes through the point $(1,\ 1)$ and is perpendicular to the line passing through points $(0,\ 2)$ and $(3,\ 3)$.

Slope of the line through points $(0,\ 2)$ and $(3,\ 3)$ is

$$\frac{3-2}{3-0} = \frac{1}{3}$$

Slope of the perpendicular line is -3. Using point-slope form:

$$y - 1 = -3(x-1) \implies 3x + y - 4 = 0$$

Equations of lines and circles – example 19

Question

Determine the general equation of the line that passes through point $(3,\ 4)$ and is parallel to the line passing through points $(0,\ -2)$ and $(3,\ 0)$.

Slope of the line through points $(0,\ -2)$ and $(3,\ 0)$ is

$$\frac{0-(-2)}{3-0} = \frac{2}{3}$$

Slope of the parallel line is the same. Using point-slope form:

$$y - 4 = \frac{2}{3}(x-3) \Longrightarrow 2x - 3y + 6 = 0$$

18. Analytic geometry - reflections and lines

Symmetric point with respect to a line

Problem

Given a point $(x_0,\ y_0)$, find its reflection with respect to the line $ax + by + c = 0$.

Let $\boldsymbol{n} = (a,\ b)$ be a normal vector to the line. The signed distance from the point to the line is:

$$\frac{ax_0 + by_0 + c}{\sqrt{a^2 + b^2}}.$$

The unit normal vector is:

$$\frac{(a,\ b)}{\sqrt{a^2 + b^2}}.$$

The vector from the point to the line is:

$$\boldsymbol{v} = \frac{ax_0 + by_0 + c}{a^2 + b^2}(a,\ b).$$

The symmetric point is:

$$(x_0,\ y_0) - 2\boldsymbol{v} = (x_0,\ y_0) - \frac{ax_0 + by_0 + c}{a^2 + b^2}(2a,\ 2b).$$

Reflections and lines – example 1

Question

Determine the point symmetric to $(3,\ -4)$ with respect to the line $-8x + 6y - 2 = 0$.

$$(3,\ -4) - \frac{-8 \times 3 - 6 \times 4 - 2}{64 + 36}(-16,\ 12) = (-5,\ 2)$$

Reflections and lines – example 2

Question

Determine the point symmetric to $(-5,\ 2)$ with respect to the line $-8x + 6y - 2 = 0$.

$$(-5,\ 2) - \frac{8 \times 5 + 6 \times 2 - 2}{64 + 36}(-16,\ 12) = (3,\ -4)$$

Reflections and lines – example 3

Question

Determine the point symmetric to $(2,\ -3)$ with respect to the line $3x - 2y + 1 = 0$.

$$(2,\ -3) - \frac{3 \times 2 + 2 \times 3 + 1}{9 + 4}(6,\ -4) = (-4,\ 1)$$

Reflections and lines – example 4

Question

Determine the point symmetric to $(-4,\ 1)$ with respect to the line $3x - 2y + 1 = 0$.

$$(-4,\ 1) - \frac{-3 \times 4 - 2 \times 1 + 1}{9 + 4}(6,\ -4) = (2,\ -3)$$

Reflections and lines – example 5

Question

Determine the point symmetric to $(3,\ 2)$ with respect to the line $4x + 6y + 2 = 0$.

$$(3,\ 2) - \frac{4 \times 3 + 6 \times 2 + 2}{16 + 36}(8,\ 12) = (-1,\ -4)$$

Reflections and lines – example 6

Question

Determine the point symmetric to $(-1,\ -4)$ with respect to the line $4x+6y+2=0$.

$$(-1,\ -4) - \frac{-4\times 1 - 6\times 4 + 2}{16+36}(8,\ 12) = (3,\ 2)$$

Reflections and lines – example 7

Question

Determine the point symmetric to $(3,\ 1)$ with respect to the line $4x + 4y = 0$.

$$(3,\ 1) - \frac{4 \times 3 + 4 \times 1}{16 + 16}(8,\ 8) = (-1,\ -3)$$

Reflections and lines – example 8

Question

Determine the point symmetric to $(-1,\ 3)$ with respect to the line $4x + 4y = 0$.

$$(-1,\ 3) - \frac{-4 \times 1 + 4 \times 3}{16 + 16}(8,\ 8) = (-3,\ 1)$$

Reflections and lines – example 9

Question

Determine the point symmetric to $(2,\ 3)$ with respect to the line $4x + 4y = 0$.

$$(2,\ 3) - \frac{4 \times 2 + 4 \times 3}{16 + 16}(8,\ 8) = (-3,\ -2)$$

Reflections and lines – example 10

Question

Determine the point symmetric to $(-2,\ -1)$ with respect to the line $4x + 4y = 0$.

$$(-2,\ -1) - \frac{-4 \times 2 - 4 \times 1}{16 + 16}(8,\ 8) = (1,\ 2)$$

Reflections and lines – example 11

Question

Determine the point symmetric to $(-2,\ -1)$ with respect to the line $4x + 4y - 20 = 0$.

$$(-2,\ -1) - \frac{-4 \times 2 - 4 \times 1 - 20}{16 + 16}(8,\ 8) = (6,\ 7)$$

Reflections and lines – example 12

Question

Determine the point symmetric to $(-2,\ -1)$ with respect to the line $4x + 4y - 4 = 0$.

$$(-2,\ -1) - \frac{-4 \times 2 - 4 \times 1 - 4}{16 + 16}(8,\ 8) = (2,\ 3)$$

Reflections and lines – example 13

Question

Determine the point symmetric to $(2,\ 3)$ with respect to the line $4x + 4y - 4 = 0$.

$$(2,\ 3) - \frac{4 \times 2 + 4 \times 3 - 4}{16 + 16}(8,\ 8) = (-2,\ -1)$$

Reflections and lines – example 14

Question

Determine the point symmetric to $(6,\ 7)$ with respect to the line $4x + 4y - 20 = 0$.

$$(6,\ 7) - \frac{4 \times 6 + 4 \times 7 - 20}{16 + 16}(8,\ 8) = (-2,\ -1)$$

Reflections and lines – example 15

Question

Determine the point symmetric to $(1,\ 2)$ with respect to the line $4x + 4y = 0$.

$$(1,\ 2) - \frac{4 \times 1 + 4 \times 2}{16 + 16}(8,\ 8) = (-2,\ -1)$$

19. System of nonlinear equations

System of nonlinear equations – example 1

Question

Solve the following system of nonlinear equations:

$$\begin{aligned} x^2 - 3y^2 - y &= 0 \\ xy - x &= 0 \end{aligned}$$

$$xy - x = 0 \Longrightarrow x(y-1) = 0 \Longrightarrow x = 0 \text{ or } y = 1$$

- For $x = 0$, substitute into the first:

$$0^2 - 3y^2 - y = -3y^2 - y = 0 \Longrightarrow y = 0 \text{ or } y = -\frac{1}{3}$$

- For $y = 1$, substitute into the first:

$$x^2 - 3 - 1 = x^2 - 4 = 0 \Longrightarrow x = \pm 2$$

System of nonlinear equations – example 2

Question

Solve the following system of nonlinear equations:

$$\begin{aligned} x^2 - 2y^2 - y &= 0 \\ xy - x &= 0 \end{aligned}$$

From the second equation:

$$xy - x = 0 \Longrightarrow x = 0 \text{ or } y = 1$$

- For $x = 0$, substitute into the first equation:

$$0^2 - 2y^2 - y = -2y^2 - y = 0 \Longrightarrow y = 0 \text{ or } y = -\frac{1}{2}$$

- For $y = 1$, substitute into the first equation:

$$x^2 - 2 - 1 = x^2 - 3 = 0 \Longrightarrow x = \pm\sqrt{3}$$

System of nonlinear equations – example 3

Question

Solve the following system of nonlinear equations:

$$\begin{aligned} x^2 - y^2 - y &= 0 \\ xy - 2x &= 0 \end{aligned}$$

From the second equation:

$$xy - 2x = 0 \implies x(y-2) = 0 \implies x = 0 \text{ or } y = 2$$

- For $x = 0$, substitute into the first:

$$0^2 - y^2 - y = -y^2 - y = 0 \implies y = 0 \text{ or } y = -1$$

- For $y = 2$, substitute into the first:

$$x^2 - 2^2 - 2 = x^2 - 4 - 2 = x^2 - 6 = 0 \implies x = \pm\sqrt{6}$$

System of nonlinear equations – example 4

Question

Solve the following system of nonlinear equations:

$$\begin{aligned} -x^2 + y^2 + x &= 0 \\ xy + y &= 0 \end{aligned}$$

$$xy + y = 0 \implies y(x+1) = 0 \implies y = 0 \text{ or } x = -1$$

- For $y = 0$, substitute into the first:

$$-x^2 + 0 + x = -x^2 + x = 0 \implies x(x-1) = 0 \implies x = 0 \text{ or } x = 1$$

- For $x = -1$, substitute into the first:

$$-(-1)^2 + y^2 - 1 = -1 + y^2 - 1 = y^2 - 2 = 0 \implies y = \pm\sqrt{2}$$

System of nonlinear equations – example 5

Question

Solve the following system of nonlinear equations:

$$\begin{aligned} x^2 - 2y^2 - y &= 0 \\ 2xy - x &= 0 \end{aligned}$$

$$2xy - x = 0 \Longrightarrow x(2y-1) = 0 \Longrightarrow x = 0 \text{ or } y = \frac{1}{2}$$

- $x = 0$ substitute into the first:

$$0^2 - 2y^2 - y = -2y^2 - y = 0 \Longrightarrow y = 0 \text{ or } y = -\frac{1}{2}$$

- $y = \frac{1}{2}$ substitute into the first:

$$x^2 - 2\left(\frac{1}{2}\right)^2 - \frac{1}{2} = x^2 - \frac{1}{2} - \frac{1}{2} = x^2 - 1 = 0 \Longrightarrow x = \pm 1$$

System of nonlinear equations – example 6

Question

Solve the following system of nonlinear equations:

$$\begin{aligned} -x^2 + 2y^2 + x &= 0 \\ xy + 2y &= 0 \end{aligned}$$

$$xy + 2y = 0 \implies y(x+2) = 0 \implies y = 0 \text{ or } x = -2$$

- $y = 0$ substitute into the first:

$$-x^2 + 0 + x = -x^2 + x = 0 \implies x(x-1) = 0 \implies x = 0 \text{ or } x = 1$$

- $x = -2$ substitute into the first:

$$-(-2)^2 + 2y^2 + (-2) = -4 + 2y^2 - 2 = 2y^2 - 6 = 0 \implies y^2 = 3 \implies y = \pm\sqrt{3}$$

System of nonlinear equations – example 7

Question

Solve the following system of nonlinear equations:

$$\begin{aligned} -3x^2 + 2y^2 - x &= 0 \\ xy + y &= 0 \end{aligned}$$

$$xy + y = y(x+1) = 0 \implies y = 0 \text{ or } x = -1$$

- $y = 0$ substitute into the first:

$$-3x^2 + 0 - x = -3x^2 - x = 0 \implies x(-3x - 1) = 0 \implies x = 0 \text{ or } x = -\frac{1}{3}$$

- $x = -1$, substitute into the first:

$$-3(-1)^2 + 2y^2 - (-1) = -3 + 2y^2 + 1 = 2y^2 - 2 = 0 \implies y^2 = 1 \implies y = \pm 1$$

System of nonlinear equations – example 8

Question

Solve the following system of nonlinear equations:

$$\begin{aligned} 3x^2 + 2y^2 - 8 &= 0 \\ xy - x &= 0 \end{aligned}$$

$$xy - x = x(y-1) = 0 \Longrightarrow x = 0 \text{ or } y = 1$$

- For $x = 0$:

$$3(0)^2 + 2y^2 - 8 = 2y^2 - 8 = 0 \Longrightarrow y^2 = 4 \Longrightarrow y = \pm 2$$

- For $y = 1$:

$$3x^2 + 2 - 8 = 3x^2 - 6 = 0 \Longrightarrow x^2 = 2 \Longrightarrow x = \pm\sqrt{2}$$

System of nonlinear equations – example 9

Question

Solve the following system of nonlinear equations:

$$\begin{aligned} x^2 + 2y^2 - 18 &= 0 \\ xy - x &= 0 \end{aligned}$$

$$xy - x = x(y-1) = 0 \Longrightarrow x = 0 \text{ or } y = 1$$

- For $x = 0$:

$$x^2 + 2y^2 - 18 = 2y^2 - 18 = 0 \Longrightarrow y^2 = 9 \Longrightarrow y = \pm 3$$

- For $y = 1$:

$$x^2 + 2 - 18 = x^2 - 16 = 0 \Longrightarrow x = \pm 4$$

System of nonlinear equations – example 10

Question

Solve the following system of nonlinear equations:

$$\begin{aligned} -2x^2 + y^2 + x &= 0 \\ xy + y &= 0 \end{aligned}$$

$$xy + y = y(x+1) = 0 \implies y = 0 \text{ or } x = -1$$

- For $y = 0$:

$$-2x^2 + 0 + x = -2x^2 + x = 0 \implies x(-2x+1) = 0 \implies x = 0 \text{ or } x = \frac{1}{2}$$

- For $x = -1$:

$$-2 + y^2 - 1 = y^2 - 3 = 0 \implies y = \pm\sqrt{3}$$

20. Logic

Logic – example 1

Question

Let the following statement be true: "If both Jana and Klára go out, then I will go out too." Which of the following statements is also logically true?

A. If I did not go out, then at least one of Jana or Klára stayed home.
B. If I went out, then both Jana and Klára went out.
C. If only Jana went out, then I must have gone out.
D. If Klára stayed home, then I did not go out.
E. If either Jana or Klára does not go out, then I will not go out.

Logic – example 1

"If Jana and Klára go out, then I will go out too." can be written symbolically as:

$$(J \wedge K) \rightarrow I$$

A. Contrapositive of the original statement: $\neg I \rightarrow (\neg J \vee \neg K)$
   Correct: logically equivalent

B. Converse of the original statement: $I \rightarrow (J \wedge K)$
   Incorrect: converse not implied

C. $(J \wedge \neg K) \rightarrow I$
   Incorrect: not covered by original statement

D. $\neg K \rightarrow \neg I$
   Incorrect: not logically implied

E. $\neg(J \wedge K) \rightarrow \neg I$
   Incorrect: inverse not implied. The original statement only tells us what happens if both Jana and Klára go out. It says nothing about what happens if one or both stay in. So, the inverse introduces a new condition and may not follow logically from the original.

Party Planning Scenario

Consider the following scenario for the subsequent examples.

Scenario

Five classmates agreed to organize a party. They discussed what snacks to buy and compiled a list of everyone's suggestions. Each classmate had specific preferences. Adam said that if there is cola and no sandwiches, then he wants either chips or strudel. Braňo wanted either chips or strudel, along with cola. Cyril stated that if there is no ice cream, then he does not want strudel either. Dano proposed having either beer or cola, but only if chips are also included. Edo wanted either sandwiches or strudel, along with beer.

Logic – example 2

Question

Edo went shopping but forgot the list at the dormitory. He bought sandwiches, beer, and strudel. Which of the remaining four classmates were satisfied with the items he brought?

- **Adam**: No cola $\Rightarrow$ the condition is false, so the implication is vacuously true $\Rightarrow$ he is satisfied.
- **Braňo**: Wanted cola, but none was bought $\Rightarrow$ not satisfied.
- **Cyril**: No ice cream $\Rightarrow$ he does not want strudel. But strudel was present $\Rightarrow$ not satisfied.
- **Dano**: Chips not included $\Rightarrow$ not satisfied.

Logic – example 3

Question

Adam went shopping but forgot the list at the dormitory. He brought sandwiches, cola, and ice cream. Which of the remaining four classmates were satisfied with the items he brought?

- **Braňo**: No chips or strudel (though cola is present) $\Rightarrow$ one requirement missing $\Rightarrow$ not satisfied.
- **Cyril**: Ice cream is present $\Rightarrow$ antecedent of his condition is false $\Rightarrow$ implication vacuously true $\Rightarrow$ satisfied.
- **Dano**: Chips not included $\Rightarrow$ prerequisite not met $\Rightarrow$ not satisfied.
- **Edo**: Wanted beer (not bought) $\Rightarrow$ not satisfied.

Logic – example 4

Question

Dano went shopping but left the list in the dormitory. He bought chips, beer, and cola. Which of the remaining four classmates were satisfied with the items he brought?

- **Adam**: There is cola and no sandwiches $\Rightarrow$ he wants chips or strudel. Chips were bought $\Rightarrow$ satisfied.
- **Braňo**: Chips or strudel and cola were bought $\Rightarrow$ satisfied.
- **Cyril**: No ice cream $\Rightarrow$ he does not want strudel. Strudel not included $\Rightarrow$ satisfied.
- **Edo**: Sandwiches or strudel with beer. Neither sandwiches nor strudel present $\Rightarrow$ not satisfied.

Logic – example 5

Question

Cyril went shopping but forgot the list at the dormitory. He brought ice cream, chips, and strudel. Which of the remaining four classmates were satisfied with the items he brought?

- **Adam**: No cola and no sandwiches $\Rightarrow$ the antecedent is false, implication vacuously satisfied $\Rightarrow$ satisfied.
- **Braňo**: No cola was bought $\Rightarrow$ not satisfied.
- **Dano**: No beer or cola $\Rightarrow$ not satisfied.
- **Edo**: No beer $\Rightarrow$ requirement (sandwiches or strudel + beer) not satisfied $\Rightarrow$ not satisfied.

Logic – example 6

Question

Adam went shopping but forgot the list at the dormitory. He brought sandwiches, cola, and ice cream. Which of the remaining four classmates were satisfied with the items he brought?

- **Braňo**: Cola is present, but neither chips nor strudel is included $\Rightarrow$ not satisfied.
- **Cyril**: Ice cream is included $\Rightarrow$ antecedent false, so implication is vacuously true $\Rightarrow$ satisfied.
- **Dano**: There is cola, but not chips $\Rightarrow$ not satisfied.
- **Edo**: Sandwiches are present, but no beer $\Rightarrow$ not satisfied.